The Beam-Like Behavior of Space Trusses

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A general approach is developed for determining the overall elastic behavior of space trusses. This is done by casting the matrix equations for pin-ended bars in a suitable algebraic form and using them to construct the finite difference equations for the truss under examination. Characteristic solutions of the homogeneous form of these equations are found in terms of simple polynomial functions. These correspond to the resultant forces and moments applied to the truss. All other modes decay toward these characteristic deflections. This means that the internal loads in the truss can be estimated from the resultant applied loading, even when the truss is not statically determinate. Also, the characteristic response yields elastic properties akin to the axial, torsional, bending, and shear stiffness of beams, although in some cases these effects are coupled. Values for a number of common space trusses are listed. It is also shown how the method may be used to give the natural frequencies of vibration of such structures.

Nomenclature

a,b,c,d = EA/ℓ^3 values for across, breadth, chord, and diagonal bars, respectively
A = cross-sectional area of a bar
E = Young's modulus
E = finite difference operator
EA = equivalent axial stiffness
$EI_{1\text{ or }2}$ = equivalent principal bending stiffnesses
f_{ij} = flexibility coefficient defined by Eq. (3)
$f_{I\text{ or }II}$ = modified end force vector given by Eq. (11)
$F_{I\text{ or }II}$ = end force vector given by Eq. (6)
F_x,F_y,F_z = end forces in the coordinate directions
GJ = equivalent torsional stiffness
h = breadth of a truss
k = modified stiffness matrix typified by Eq. (13)
K = stiffness matrix defined by Eq. (8)
ℓ = length of a bar
m = mass at a joint
$M_{1\text{ or }2}$ = moments about the principal axes
P = axial force
s = span of a bay
$S_{1\text{ or }2}$ = shear forces in the directions of the principal axes
T = torque
u,v,w = end displacements in the directions of the coordinates
$u_{I\text{ or }II}$ = end displacement vector given by Eq. (7)
$U_{I\text{ or }II}$ = modified end displacement vector given by Eq. (12)
x,y,z = Cartesian coordinate lengths
X = nondimensional coordinate x/s
α = anticlockwise angle of the principal axes to the y and z axes
$\gamma_{1\text{ or }2}$ = displacement rate through which $S_{1\text{ or }2}$ does work
ϵ = tensile axial strain
θ = rate of twist
ϕ_1,ϕ_2 = rates of rotation about the principal axes
ω = natural frequency of vibration

Introduction

THE ability to analyze trusses in a manner analogous to that for beams provides the designer with a means of estimating the performance of structures prior to the detailed examination of a particular design. As noted by Noor et al.,[1] several methods have already been employed. One is to make a direct analogy between the stiffness of members of the truss in particular directions and the stiffness of an equivalent continuum. This suffers from the defect that the role played by the truss members under load may be unrelated to the behavior of the continuum under an analogous load. For example, there may be bars that, because of the configuration of the truss, carry no forces when they are loaded at the ends only; an instance of this will be given later. Such bars cannot contribute to the stiffness characteristics of the truss in relation to such loading.

Noor et al.[1] use an energy method in which the nodal displacements of the truss are related to a linearly varying displacement field for an equivalent bar. This method is only applied to trusses with triangular cross sections where the nine parameters fully describe the displacements. Exact analytic solutions[2,3] of the behavior of trusses under load have been presented using finite difference calculus. Expressing the operators in terms of Taylor's series allows continuum approximations to the finite difference equations to be made, resulting in equivalent plate stiffnesses, for example.[4] It has not proved possible until now to extend this kind of approach to give equivalent beam properties other than the bending stiffness. This has been remedied by examining afresh how such properties can be derived.

In normal engineering analyses, the bending of beams is determined from the resultant bending moment, their torsion from the resultant torque, and their extension from the resultant axial force, no matter how these loads are applied. This behavior can be derived from the exact characteristic modes induced by bending moments, torques, and axial forces. These modes are uniform along the beam, so that the strains are constant along the beam but may vary in a more complex fashion across it. Since the strains do not vary longitudinally, the only displacement variations between sections are linearly varying rigid-body motions corresponding to flexure, torsion, extension, and shear.

For isotropic prismatic beams, these effects are uncoupled, and shear only occurs under the action of shear forces in a manner to be discussed more fully later. This uncoupling can be seen to result from symmetry. For example, consider the prismatic beam shown in Fig. 1. The torque T and moment M, produce rotations θ and ϕ due to the twist and flexure of the beam. The mirror image produced by mapping the points (x,y,z) onto $(-x,y,z)$ is shown in Fig. 1b. If the bar is isotropic as well as prismatic, the image is identical to the original form, although the sense of T and θ has changed.

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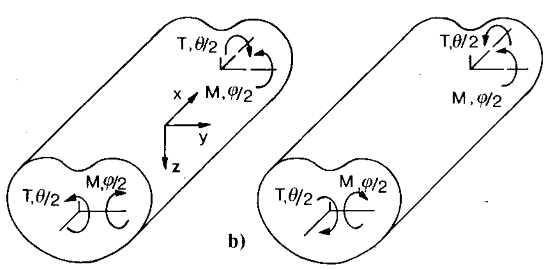

Fig. 1 Symmetry and coupling for a prismatic bar.

Combining the two cases gives flexure alone under the action of a moment $2M$ and taking the difference of the two cases gives torsion alone under the action of a torque $2T$. This uncoupling does not necessarily occur if the material is anisotropic. The same image results from inversion of the y axis, provided that the beam and its material properties are symmetrical with respect ot the xz plane. Thus, under this condition uncoupling will again occur, even if the beam is not prismatic. However, it would be wrong to assume that flexure and torsion were uncoupled for a tapered wing section for example, even if it were made from an isotropic material. For the space trusses examined here, uncoupling is only assured if such symmetry conditions apply.

In this analysis, a more general comparison will be made with anisotropic beams, using the concept of characteristic modes. Under the action of constant torques, bending moments, and axial forces, these modes correspond to a set of bar strains which are the same for each module of the truss. This produces relative rigid-body deflections between modules of the truss which vary linearly along the truss, corresponding to torsion, flexure, extension, and shear. However, the action of a shear force S necessarily produces a linearly varying bending moment. The characteristic bending-shear mode must then give linearly varying strains in at least some of the bars. From this mode, the complementary energy per unit length C can be determined. At a distance x from the end at which S is applied, this takes the form

$$C=\tfrac{1}{2}f_m(Sx)^2+f_cS^2x+\tfrac{1}{2}f_sS^2 \tag{1}$$

where, since C is positive-definite, the flexibility coefficients satisfy the conditions

$$f_m>0,\quad f_s>0,\quad f_mf_s-f_c^2>0 \tag{2}$$

When a constant bending moment only is applied, f_m times the moment gives the curvature. In the case of uncoupled equations, f_m is the inverse of the corresponding bending stiffness. In all of the cases examined here, the coupling flexibility f_c is zero. The shear flexibility f_s times S gives the displacement per unit length γ through which S does work, purely as a shear load (i.e., it is an effective shear per unit length). Making the usual assumptions of plane stress for a rectangular isotropic beam, this method gives a value of $1.2/GA$ for f_s (see, for example, Washizu[5]) where G is the shear modulus and A is the cross-sectional area of the beam. Uniform shear, which cannot occur without applying shear stresses to the sides of the beam, gives a value of $1/GA$ for f_s. The above energy definition of shear flexibility is not the only possible one. For the same rectangular beam, Timoshenko and Goodier[6] estimate it as $1.5/GA$, Lowe[7] gives a value of $1/GA(1-\nu^2)$, and Donnell[8] derives a value of $3(8+5\nu)/20GA(1+\nu)$ where ν is Poisson's ratio. For the space trusses, it would also be possible to obtain a value for the shear flexibility from the relationship between the shear force and the rotation of some mean cross section through the joints relative to some mean lateral deflection of the joints. However, the flexibility f_s^* found in this way is not necessarily positive-definite. The reason for this will be discussed later.

The values quoted in the Appendix are obtained from the mean complementary energy per unit length, taking the midpoints of typical members to be at a distance x along the truss. In accordance with Eq. (2), these are positive-definite.

The results in the Appendix are in terms of flexibility coefficients, because it is usually the deflections produced by known loads that are sought rather than the converse. The relationships are given by the general form

$$\begin{bmatrix}\theta\\ \phi_1\\ \phi_2\\ \epsilon\\ \gamma_1\\ \gamma_2\end{bmatrix}=\begin{bmatrix}f_{11}&f_{12}&f_{13}&f_{14}&0&0\\ f_{21}&f_{22}&0&f_{24}&f_{25}&f_{26}\\ f_{31}&0&f_{33}&f_{34}&f_{35}&f_{36}\\ f_{41}&f_{42}&f_{43}&f_{44}&f_{45}&f_{46}\\ 0&f_{52}&f_{53}&f_{54}&f_{55}&f_{56}\\ 0&f_{62}&f_{63}&f_{64}&f_{65}&f_{66}\end{bmatrix}\begin{bmatrix}T\\ M_1\\ M_2\\ P\\ S_1\\ S_2\end{bmatrix} \tag{3}$$

where from Betti's reciprocal theorem f_{ij} and f_{ji} are equal. The torque T about the longitudinal axis x of the truss produces a clockwise rate of increase of module rotation θ about this axis. M_1 is the moment about the first principal axis producing a rate of increase of rotation ϕ_1 clockwise about this axis. M_2 and ϕ_2 are similarly related to the second principal axis. The principal axes are chosen so that f_{23} (and f_{32}) are zero. These moments and rotations are shown in the positive sense in Fig. 1a. The forces P, S_1, and S_2 produce an extension, ϵ, and shears γ_1 and γ_2 corresponding to rates of increase of displacement along the truss in the directions of the longitudinal and principal axes. The nature of shear will be discussed more fully later. The absence of a coupling flexibility means that f_{26}, f_{35}, (f_{62} and f_{53}) are zero. The lines of action of S_1 and S_2 can be modified by the addition of a pure torque. They are, therefore, defined so that f_{15}, f_{16} (f_{51} and f_{61}) are zero; that is, the shear forces produces no torsion. However, torsion-like effects can still arise as in the case of the double-bay, single-laced beam listed in the Appendix and shown in Fig. 4. In this case, the shear force S_1 produces equal and opposite rates of twist of the two types of bay.

The corresponding stiffness matrix is given by the reciprocal of the flexibility matrix in Eq. (3). Where the equations are coupled, the expressions for the stiffness coefficients are generally more complex than those for the flexibility coefficients. When the equations are uncoupled, the flexibility coefficients f_{11}, f_{22}, f_{33}, and f_{44} can be expressed as the reciprocals of equivalent stiffnesses GJ, EI_1, EI_2, and EA without ambiguity. Since this notation is more widely understood, it will be used where appropriate in the Appendix.

Stiffness Matrices

For the purposes of this paper, all the members of the truss will be taken as pin-ended bars. Figure 2 shows a typical bar I-II with length ℓ, cross-sectional area A, and Young's modulus E. The projections of the length in the directions of a global coordinate system are taken to be x, y, and z, the end forces in the same directions F_x, F_y, and F_z, and the end displacements u, v, and w, respectively. The stiffness matrices for the bar can now be written as

$$F_I=Ku_I-Ku_{II} \tag{4}$$

$$F_{II}=-Ku_I+Ku_{II} \tag{5}$$

where

$$F_{I\text{ or }II}=\begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix}_{I\text{ or }II} \tag{6}$$

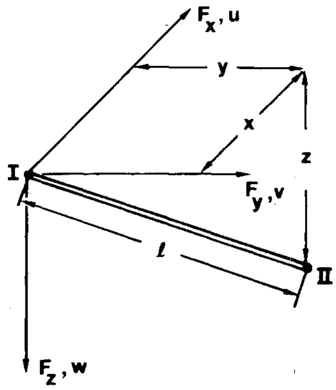

Fig. 2 The bar coordinate system.

$$u_{I\text{ or }II}=\begin{bmatrix} u \\ v \\ w \end{bmatrix}_{I\text{ or }II} \tag{7}$$

$$K=\frac{EA}{\ell^3}\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \tag{8}$$

The desired results are expressed in a general algebraic form that requires considerable manipulation of the elements of such stiffness matrices, and the above standard form is not the most convenient for this purpose. The equations can be rewritten as

$$f_I = kU_I - kU_{II} \tag{9}$$

$$f_{II} = -kU_I + kU_{II} \tag{10}$$

where

$$f_{I\text{ or }II}=\begin{bmatrix} F_x/x \\ F_y/y \\ F_z/z \end{bmatrix}_{I\text{ or }II} \tag{11}$$

$$U_{I\text{ or }II}=\begin{bmatrix} ux \\ vy \\ wz \end{bmatrix}_{I\text{ or }II} \tag{12}$$

$$k=\frac{EA}{\ell^3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{13}$$

The global coordinate systems will be chosen so that the x axis lies along the axis of the truss and the y axis is parallel to a bar lying across the breadth of the truss. The span of bays along the truss will be denoted by s and the length of the above bar across the truss denoted by h. With one exception, the configuration of joints for the space trusses analyzed here is shown in Fig. 3. For the trusses of square section, x, y, and z in Eqs. (11) and (12) will be replaced by s, h and h, respectively, and for those of triangular section by s, $h/2$, and $\sqrt{3}\,h/2$, respectively. The true projected lengths of every bar are then simple integer multiples of these values. The modified form taken by k is given by multiplying the appropriate rows and columns of the matrix in Eq. (13) by such multiples. The value of EA/ℓ^3 for "across" bars AC or BD etc., will be denoted by a; for "breadth" bars AB, BC, CD, DA, etc., by b; for "chord" bars AA', BB', CC', and DD', etc., by c, and for "diagonal" bars, AB', $A'B$, BC', $B'C$, CD', $C'D$, DA', and $D'A$, etc., by d. Typical stiffness matrices k for the trusses of square section are then for AC,

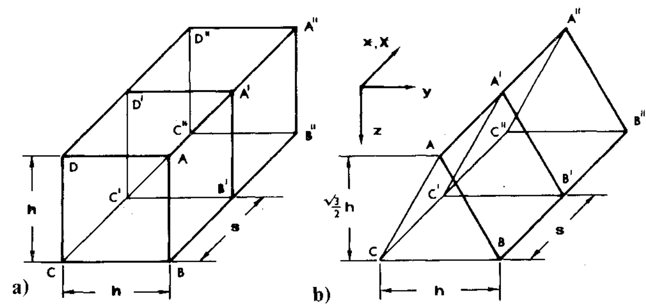

Fig. 3 Basic truss systems.

$$k=a\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{14}$$

for AB and CD,

$$k=b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

for all chord members,

$$k=c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{16}$$

for AB' and DC',

$$k=d\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{17}$$

These stiffness matrices can now be assembled into the stiffness matrix for the complete structure and solutions for different loading conditions found.

Applications of the Finite Difference Calculus Method

Figure 3 shows the types of space truss that will be examined. In addition to the bars shown, they will also have various configurations of diagonal bars and, in the case of square trusses, various combinations of across bars. The trusses consist of a series of identical modules joined end to end. If each bay is the same as every other, then the module consists of one bay. If alternate bays are identical , as in Fig. 4, then the module consists of two bays. The joints are numbered so that the Xth joint from some origin is at a distance x equal to Xs from that end, where s is the span of a bay. The deflections can then be expressed as functions of X. The operator E is such that its effect on any function $G(X)$ is given by

$$EG(X)=G(X+1) \qquad E^{-1}G(X)=G(X-1) \tag{18}$$

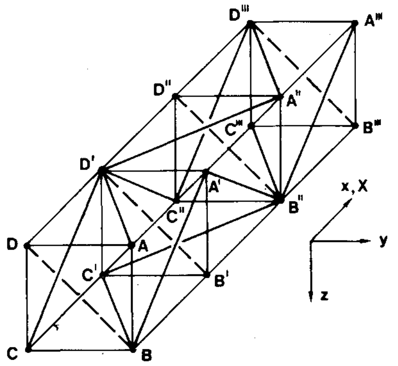

Fig. 4 Double-bay, single-laced beam (SPAS system).

If the truss module consists of a single bay, these operations enable the deflections of similar joints either side of the Xth joint to be expressed in terms of those at the Xth joint. In the case of a two-bay module, as in Fig. 4, joints A and A''' are similar, and joints A' and A''' are similar but different from the first pair. Taking A' to be X along the truss, the displacements of A''' in the x, y, and z directions, u_A''', v_A''', w_A''' are given in terms of those at A', u_A', w_A' by

$$u_A''' = E(Eu_A') = E^2 u_A', \quad v_A''' = E^2 v_A', \quad w_A''' = E^2 w_A' \tag{19}$$

It is convenient to define the displacements of A and A'' in terms of an imaginary joints $\bar{A}$ of the same type, X along the truss. Then

$$u_A = E^{-1}\bar{u}_A, \quad v_A = E^{-1}\bar{v}_A, \quad w_A = E^{-1}\bar{w}_A \tag{20}$$

$$u_A'' = E\bar{u}_A, \quad v_A'' = E\bar{v}_A, \quad w_A'' = E\bar{w}_A \tag{21}$$

Similar expressions hold for the other joints.

Using the stiffness matrices of the previous section, stiffness equations can be formed, relating the loads applied to the joints of a typical module to the joint deflections of that module and the adjoining modules. Using the operator E, all the deflections can then be expressed in terms of those of the typical module. The characteristic modes are given by deflection functions which are finite polynomials of X. Apart from rigid-body motions, there are six such modes, related to torsion, bending, extension, and shear. These satisfy the homogeneous form of the stiffness equations, that is, there are no loads on the joints between the ends of the truss. Such solutions apply to a truss which could stretch ad infinitum in either direction. The boundary conditions imposed on a finite truss will not necessarily correspond to those which could be transmitted by a virtual extension of the truss beyond its boundaries. In this sense the solutions are incomplete, but the necessary corrections will apply to the end bays only. Such corrections have been used in solutions for plane trusses.[3] The total number of independent solutions of the homogenous form of the equations, other than rigid-body motions, can be found by sectioning through the truss at typical bays near its ends. The total number of statically independent bar forces at these sections gives the total number of independent modes of deformation between the sections. For example, sectioning through a typical bay of the double-bay single-laced beam shown in Fig. 4 yields eight independent bar forces. Sectioning through two end bays of such a truss indicates 16 bar forces acting on the truss between, but there are only 10 statically independent bar forces, allowing for the six equations of statics which relate them. This means that in addition to the six characteristic modes, there are a further four independent solutions of the homogeneous case, other than rigid-body deflections. For the truss shown, these are

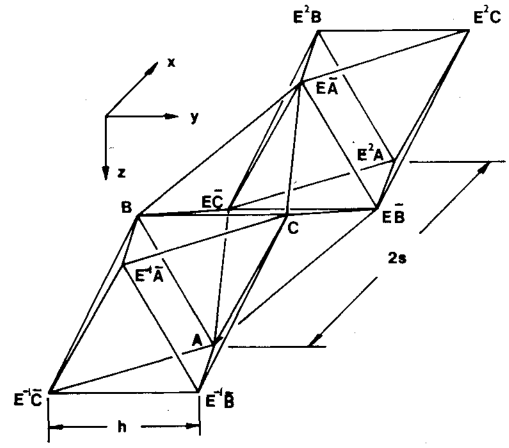

Fig. 5 Inverted batten beam.

typified by deflections of the form

$$u = KR^{-X}\cos(\beta X + \mu) \tag{22}$$

where

$$Re^{i\beta} + \frac{1}{R}e^{-i\beta} = 2 + \frac{1}{cd(2a+b)}\{ab(2c-d) \pm \sqrt{a^2b^2(2c-d)^2 - 16abcd^2(2a+b)}\} \tag{23}$$

This gives a pair of real reciprocal roots for R and equal and opposite values for β. The value of R greater than unity gives a pair of solutions decaying from the negative (low X) end of the truss and its reciprocal gives a pair decaying back from the positive (high X) end of the truss. To give an idea of the rate of decay, suppose that all the truss members have the same cross section and the span of a bay is equal to its breadth. Then

$$R = (3.197)^{\pm 1}, \quad \beta = \pm 0.1798\pi \tag{24}$$

This means that any difference in the applied loading at the ends from that given by the characteristic solutions with the same resultants decays by a factor of slightly more than 10 in the space of one module (two bays). In the particular case where either a or b is zero, there is no exponential decay. This indicates the importance of cross-bracing if a beam-like analysis to to be used. Hoff[9] has examined the behavior of a space truss of square cross section where no cross-bracing ($a=0$) is used between the ends. In this case, the bar forces produced by a self-equilibrating end loading do not decay exponentially but are either constant or vary linearly along the truss. The characteristic modes, which are related to the resultant loading, would then be insufficient to describe the overall behavior of the truss. The effect of the cross-bracing is to insure that not only the complete structure but also each module of the structure is not a mechanism. If the modules are not mechanisms, they will resist all but rigid-body rotations and displacements, tending to absorb rather than transmit loading systems which have no resultant moment or force, thus causing the decay of such systems along the truss. St. Venant's principle can then be applied, permitting an approximate beam analysis to be used to characterize the response of the truss.

It should be noted that, in the absence of intermediate joint loading, the forces in bars of the type AB, BC, $A'D'$, and $C'D'$ of the truss in Fig. 4 are necessarily zero. Any method of determining the stiffness of an equivalent continuum, which is simply based on the material used per unit area, is then likely to give misleading answers, as the role played by the above bars in the general behavior of the truss is negligible.

Flexure of an Inverted Batten Beam

As an example of the way in which the method is applied, a subset of the solutions for the inverted batten beam shown in Fig. 5 will be examined. This is obtained from the subset of equations which result on taking

$$u_A = -\bar{u}_A, \quad v_A = \bar{v}_A = 0, \quad w_A = \bar{w}_A$$

$$u_B = u_C = -\bar{u}_B = -\bar{u}_C, \quad v_B = -v_C = \bar{v}_B = -\bar{v}_C$$

$$w_B = w_C = \bar{w}_B = \bar{w}_C \tag{25}$$

This assumption satisfies the homogenous form of the original set of 18 finite difference equations, provided that

$$\begin{bmatrix} 2d & 0 & d(E^{-1}+E) & d(E^{-1}-E) & -d(E^{-1}-E) \\ 0 & 9b+2d & d(E^{-1}-E) & -3b+d(E^{-1}+E) & -9b-d(E^{-1}+E) \\ d(E+E^{-1}) & d(E-E^{-1}) & d(4+E^{-1}+E) & 0 & 2d(E^{-1}-E) \\ d(E-E^{-1}) & -3b+d(E+E^{-1}) & 0 & 9b+2d & 3b-2d \\ -d(E-E^{-1}) & -9b-d(E+E^{-1}) & 2d(E-E^{-1}) & 3b-2d & 9b+d(10-4E^{-1}-4E) \end{bmatrix} \begin{bmatrix} U_A \\ W_A \\ U_B \\ V_B \\ W_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

where

$$U_A = su_A, \quad W_A = \frac{h}{2\sqrt{3}} w_A, \quad U_B = su_B,$$

$$V_B = \tfrac{1}{2} h v_B, \quad W_B = \frac{h}{2\sqrt{3}} w_B \tag{27}$$

Omitting a constant multiplier, a solution of these equations takes the form

$$U_A = -4X, \quad W_A = X^2, \quad U_B = 2X, \quad V_B = g, \quad W_B = X^2 - g \tag{28}$$

where

$$g = 3d/(3b+2d) \tag{29}$$

This solution gives equal strains in all bars of the same type, producing rotation of modules about the y axis at a uniform rate along the truss (flexure). The bar forces produced by these strains have the same resultant at each section, a moment about the y axis. Relating this moment to the flexural curvature gives a bending stiffness

$$EI_I = \frac{1}{f_{22}} = \frac{9h^2 s^3 bd}{4(3b+2d)} \tag{30}$$

Differentiating the solution given by Eq. (28) gives the form

$$U_A = -2, \quad W_A = X, \quad U_B = 1, \quad V_B = 0, \quad W_B = X \tag{31}$$

which is a simple rigid-body rotation of the whole frame about the y axis. There also exists an integral of the solution given by

$$U_A = -6X^2 - 2\psi, \quad W_A = X^3, \quad U_B = 3X^2 + \psi,$$

$$V_B = 3gX, \quad W_B = X^3 - 3gX \tag{32}$$

where

$$\psi = \frac{2(6b-5d)}{3b+2d} \tag{33}$$

The bar forces given by this mode produce a resultant shear force S_2 and a linearly varying bending moment $S_2 sX$ at any section. It will be seen from the expressions for U_A and U_B that the cross section through A, B, and C has rotated relative to the mean deflection curve given by the average of w_A, w_B, and w_C. Relating this to the shear force, S_2 gives an apparent shear flexibility

$$f_{66}^* = \frac{8(3b-d)}{27bdsh^2} \tag{34}$$

It should be noted that this is not positive-definite. However, changes in deflection are not meaningful for increments in x of less than the span s. Applying the results given by Eqs. (30) and (34) to a cantilever of length s with a shear force S_2 at the free end gives the free-end displacement as

$$w = \frac{1}{3} f_{22} S_2 s^3 + f_{66}^* S_2 s = \frac{4S_2}{3h^2 d} \tag{35}$$

Thus, over this minimum length, and over all greater lengths, there is a positive displacement in the direction of S_2.

The matrices described earlier readily yield the bar tension coefficients t from the deflections given by Eq. (32). These can be used to determine the shear force

$$S_2 = -\frac{27\sqrt{3}bdh}{3b+2d} \tag{36}$$

and the complementary energy per unit length is given by the sum of $\tfrac{1}{2}t^2/(EA/\ell^3)$ for all the typical members of the truss, divided by $2s$,

$$C = \frac{243bd}{2s(3b+2d)^2}\left[X^2(12b+8d) + 9b\right] \tag{37}$$

Comparing Eqs. (36) and (37) with Eq. (1) yields the same value for f_{22} as that given by Eq. (30) and gives the shear stiffness as

$$f_{66} = 1/dsh^2 \tag{38}$$

The values of the shear stiffnesses given in the Appendix are found in this way.

The solution given by Eq. (28) can be thought of as being in terms of a single parametric function w where

$$U_A = -2sw', \quad W_A = w, \quad U_B = sw'$$

$$V_B = \tfrac{1}{2} gs^2 w'', \quad W_B = w - \tfrac{1}{2} gs^2 w'' \tag{39}$$

where the primes indicate differentiation with respect to x and, in this case, w is X^2. This form gives a rigid-body displacement in the z direction. All of the terms in Eq. (32), except those in ψ, are given on taking w as X^3. It will be seen that ψ is related to a rotation of the cross section and can be taken as a second parametric function related to shear. This approach can be applied to the other space trusses, but in the case of the square trusses, the terms in addition to the bending parametric function, w, do not necessarily represent a rotation of a section through the joints. The significance of this becomes apparent when applying the Timoshenko beam theory to the vibration of trusses. This allows for a rotation of the cross section as well as a displacement, treating the section as a rigid plane. The natural frequencies of vibration are found from a pair of simultaneous differential equations in terms of two parameteric functions equivalent to w and ψ. This necessarily gives rise to only two natural frequencies of vibration in a given plane of a particular wavelength. However, if the behavior of a section through the joints is significantly different to that of a rigid plane, the Timoshenko beam theory is no longer appropriate. Instead, the potential energy stored in the bars per unit length can be expressed in terms of w'' and ψ, and the kinetic energy per unit length expressed in terms of $\dot{w}$, $\dot{w}'$, $\dot{w}''$, and $\dot{\psi}$, where the dot superscript indicates differentiation with respect to time. Hamilton's principle can then be applied to the Lagrangian function $L(q_i, q_i'', \dot{q}_i, \dot{q}_i', \dot{q}_i'')$, where the generalized coordinates q_1 and q_2 are w and ψ. This gives rise to a pair of differential equations equivalent to those of the Timoshenko beam theory, but without having made assumptions of the type mentioned above.

The original finite-difference equations can be modified to analyze vibration problems. In the simple case when the inertia of the truss is represented by equal masses, m, at each joint and the frame undergoes a harmonic vibration of frequency, ω, the right-hand side of Eq. (26) becomes

$$\begin{bmatrix} pU_A \\ 3rW_A \\ 2pU_B \\ 2rV_B \\ 6rW_B \end{bmatrix}, \quad p=\omega^2 m/2s^2, \quad r=2\omega^2 m/h^2 \tag{40}$$

Taking

$$U_A = U_A^0 \cos kX e^{i\omega t}, \quad W_A = W_A^0 \sin kX e^{i\omega t}$$
$$U_B = U_B^0 \cos kX e^{i\omega t}$$
$$V_B = V_B^0 \sin kX e^{i\omega t}, \quad W_B = W_B^0 \sin kX e^{i\omega t} \tag{41}$$

and solving the modified form of Eq. (26) as an eigenvalue problem gives five values for ω^2, which are the roots of

$$(C^2 - AF)[D^2 I + E^2 G - 2DHE + B(H^2 - GI)] + S^2[A(H^2 - GI) + 2C(DI - EH) - BFI + FE^2 + S^2 I] = 0 \tag{42}$$

where

$$A = 2d - p, \quad B = 9b + 2d - 3r, \quad C = 2d\cos k, \quad D = 2d\cos k - 3b$$
$$E = 6b + 4d - 6r, \quad F = (4 + 2\cos k)d - 2p, \quad G = 9b + 2d - 2r$$
$$H = 6b + 4d\cos k - 2r, \quad I = 12b + (16 - 8\cos k)d - 20r,$$
$$S = 2d\sin k \tag{43}$$

Table 1 Nondimensional natural frequencies ($\omega^2 m/s^2 d$) found from Eq. (42)

π/k	1	2	3	4	5
10	$1.427_{10^{-3}}$	1.773	2.840	2.990	7.831
30	$1.799_{10^{-5}}$	1.960	2.552	2.999	7.981
50	$2.336_{10^{-6}}$	1.985	2.520	3.000	7.993

Table 2 Percentage variation from the first two columns of Table 1 using three approximations

	Ref. 1		Eqs. (30) and (38)		Hamilton's principle	
π/k	1	2	1	2	1	2
10	+6.80	+20.1	−3.44	+19.6	−3.12	+14.6
30	+10.6	+2.76	−0.42	+2.71	−0.36	+2.20
50	+10.9	+1.03	−0.15	+1.01	−0.13	+0.83

The values of $\omega^2 m/s^2 d$ thus found are given in Table 1 for b/d and h/s equal to one. These are for half wavelengths of 10, 30, and 50 bays. Using this method, any rounding-off errors are not affected by the number of bays considered. As the wavelength increases, the second to fifth values tend to 2, 2.5, 3, and 8. Table 2 shows the percentage variation from the first two values using approximate analyses. The first four columns are derived from the Timoshenko beam theory, in the first pair of columns using the elastic constants of Noor et al.[1] and in the second pair of columns using the elastic constants of Eqs. (30) and (38). The shear constants are the same in both cases. Using the value given by Eq. (34) gives a better estimate of the first natural frequency but a much poorer estimate of the second. The third pair of columns gives the percentage error when Hamilton's principle is used. The errors and the most accurate approximation can vary with different choices of b/d, h/s, and π/k, but the second and third methods converge on the exact values as π/k increases, but this is not the case with the lowest root found by the first method. The third to fifth frequencies in Table 1 do not correspond to modes which could be deduced from a beam-like analysis. More generally, the original truss has six typical joints, each with three degrees-of-freedom, so that there can be as many as 18 separate natural frequencies corresponding to any particular wavelength. If accurate values of all the frequencies are required, then the finite-difference calculus method will provide them. If only the lowest frequencies are sought, they may be found using a beam-like approximation. This involves using a continuum approximation as well as the approximation involved in representing that continuum by a limited set of parametric functions, so that, as shown in Table 2, the error is not always positive.

Concluding Remarks

From exact analyses of the response of space trusses, it has been found that they can behave rather like ordinary beams, although sometimes they may have more unexpected characteristics. Any analysis which assumes a priori that these trusses are analogous to ordinary beams is then likely to be misleading, However, a method of determining beam-like properties is outlined in the paper and some of these properties are given. Since these are found from exact simple polynomial solutions, which exist for any resultant force or moment acting on the truss, no beam-like assumptions, such as plane sections remaining plane, are required. This permits a wider range of trusses to be examined than those analyzed previously. The method also gives an estimate of the internal bar forces in terms of the resultant forces acting on the section, even when these forces cannot be found from statics. The finite-difference equations used can be modified[10] to solve the static stability of space trusses. It has been shown that they may also be modified to give the natural frequencies of the trusses. Some of these frequencies can be found by using equivalent beam analyses. The generality of the method has also made it possible to find the beam-like properties of rigid-jointed trusses.

Appendix

Beam-Like Flexibilities of Space Trusses

The elastic constants for the trusses are expressed in terms of the EA/ℓ^3 values for across, breadth, chord, and diagonal bars, a, b, c, and d, respectively. The flexibility coefficients are defined by Eq. (3). Where no ambiguity would arise, the values of the equivalent beam stiffnesses GJ, EI_1, EI_2, and EA are given. If the principal axes 1 and 2 are not in the directions of y and z, respectively, the anticlockwise rotation α of these axes from these directions is given. The breadth and chord members shown in Fig. 2, AB and AA', for example, will be taken as common to all the trusses. The inverted batten beam is of a different type and is shown in Fig. 5. It is taken to consist of breadth and diagonal bars. All other bars in a typical module are listed below. Such a module may contain either one or two bays.

Triangular Trusses

Single-Bay, Single-Laced Beam

Diagonal bars: AB', BC', CA' (the diagonal bars form a clockwise spiral along the truss).

$$f_{11}=\frac{4}{h^4 s}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),\quad f_{14}=-\frac{2}{\sqrt{3}ch^2s^2}$$

$$1/f_{22}=1/f_{33}=EI_1=EI_2=\tfrac{1}{2}h^2s^3c$$

$$f_{25}=f_{36}=\frac{1}{\sqrt{3}ch^2s^2},\quad f_{44}=\frac{1}{3s^3c}$$

$$f_{55}=f_{66}=\frac{2}{3h^2s}\left(\frac{1}{b}+\frac{1}{4c}+\frac{1}{d}\right)$$

Single-Bay, Double-Laced Beam

Diagonal bars: AB', $A'B$, BC', $B'C$, CA', $C'A$ (the diagonal bars cross-brace the faces).

$$1/f_{11}=GJ=\tfrac{1}{2}h^4sd$$

$$1/f_{22}=1/f_{33}=EI_1=EI_2=\tfrac{1}{4}h^2s^3\left(2c+\frac{bd}{b+2d}\right)$$

$$1/f_{44}=EA=3s^3\left(c+\frac{2bd}{b+2d}\right),\quad f_{55}=f_{66}=\frac{1}{3dh^2s}$$

These results agree with those given in Table 1 of Ref. 1.

Double-Bay, Single-Laced Beam

Diagonal bars: AB', BC', CA', $A'C''$, $B'A''$, $C'B''$ (the diagonal bars spiral in opposite directions in alternate bays).

$$1/f_{11}=GJ=\frac{h^4scd}{4(c+d)}$$

$$1/f_{22}=1/f_{33}=EI_1=EI_2=\tfrac{1}{2}h^2s^3c$$

$$1/f_{44}=EA=3s^3c,\quad f_{55}=f_{66}=\frac{1}{6h^2s}\left(\frac{1}{c}+\frac{4}{d}\right)$$

These results agree with those given in Table of Ref. 1.

Inverted Batten Beam

(This is shown in Fig. 5.)

$$1/f_{11}=GJ=\tfrac{1}{2}h^4sd$$

$$1/f_{22}=1/f_{33}=EI_1=EI_2=\frac{9h^2s^3bd}{4(3b+2d)}$$

$$1/f_{44}=EA=\frac{54s^3bd}{9b+2d},\quad f_{55}=f_{66}=\frac{1}{h^2sd}$$

Apart from the torsional and shear constants, these results differ slightly from those given in Table 1 of Ref. 1.

Square Trusses

Single-Bay, Single-Laced Beam

Diagonal bars: AD', BA', CB', DC' (the diagonal bars form an anticlockwise spiral along the truss).

1) Across bar: BD

$$f_{11}=\frac{1}{h^4s}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),\quad f_{14}=\frac{1}{2h^2s^2c}$$

$$f_{22}=f_{33}=\frac{1}{h^2s^3c}$$

$$f_{25}=f_{36}=\frac{-1}{2ch^2s^2},\quad f_{44}=\frac{1}{4s^3c}$$

$$f_{55}=f_{66}=\frac{1}{2h^2s}\left(\frac{1}{b}+\frac{1}{2c}+\frac{1}{d}\right)$$

2) Across bars: AC, BD

$$f_{11}=\frac{1}{h^4s}\left(\frac{1}{2a+b}+\frac{1}{c}+\frac{1}{d}\right)$$

All other flexibility coefficients are the same as in case 1.

Single-Bay, Single-Laced Beam

Diagonal bars: AB', CB', DA', DC', $\alpha=45$ deg (the diagonals in opposite faces are parallel). Across bar: BD.

$$f_{11}=\frac{1}{h^4s}\left(\frac{1}{b}+\frac{1}{2c}+\frac{1}{d}\right),\quad f_{13}=\frac{1}{\sqrt{2}ch^3s^2}$$

$$1/f_{22}=1/f_{33}=EI_1=h^2s^3c$$

$$f_{44}=\frac{1}{4s^3c},\quad f_{46}=-\frac{1}{2\sqrt{2}chs^2}$$

$$f_{55}=\frac{1}{2h^2s}\left(\frac{1}{b}+\frac{1}{d}\right),\quad f_{66}=\frac{1}{2h^2s}\left[\frac{a+b}{(2a+b)^2}+\frac{5}{4c}+\frac{1}{d}\right]$$

Single-Bay, Double-Laced Beam

Diagonal bars: AB', $A'B$, BC', $B'C$, CD', $C'D$, DA', $D'A$.

Across bars: AC, BD (All the panels of the truss are cross-braced.)

$$1/f_{11}=GJ=8h^4sd$$

$$1/f_{22}=1/f_{33}=EI_1=EI_2=h^2s^3\left(c+\frac{bd}{b+2d}\right)$$

$$1/f_{44}=EA=4s^3\left(c+\frac{2(2a+b)d}{2a+b+2d}\right),\quad f_{55}=f_{66}=\frac{1}{4h^2sd}$$

Double-Bay, Single-Laced Beam

Diagonal bars: AD', BA', BC', CD', $A'B''$, $C'B''$, $D'C''$, $D'A''$, $\alpha = 45$ deg.

Across bars: BD, $B'D'$ (this truss, which is shown in Fig. 4, is the basis of the West German SPAS system).

$$1/f_{11} = GJ = \frac{4h^3 s^2 cd}{2c+d}$$

$$1/f_{22} = EI_1 = h^2 s^3 \left(c + \frac{abd}{2ab+2bd+4da}\right)$$

$$1/f_{33} = EI_2 = h^2 s^3 c, \quad 1/f_{44} = EA = 4s^3 c$$

$$f_{55} = \frac{1}{2h^2 sd},$$

$$f_{66} = \frac{1}{2h^2 s}\left[\frac{1}{c} + \frac{1}{d} + \frac{1}{c}\left(\frac{abd}{abd+2abc+2bcd+4acd}\right)^2\right]$$

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